Math 306 Matrices & Power Series - Theorems and Definitions

Basic Conventions: The term series refers to an infinite sum e.g. $\sum_{n=0}^{\infty} a_n$ while the term sequence refers to the terms in a sequence e.g. a_n for $n \geq 1$. Let S_N denote the Nth partial sum of a series i.e. $S_N = \sum_{n=0}^{N} a_n$, while $S = \lim_{N\to\infty} S_N = \sum_{n=0}^{\infty} a_n$.

Taylor Series: The Taylor series centered at $x = a$ of a function f is defined as

$$f(x) = \lim_{N\to\infty} T_N(x) = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Let $R_N(x) = f(x) - T_N(x)$. Then

$$|R_N(x)| \leq B\frac{|x|^{N+1}}{(N+1)!},$$

where B is a real number such that $|f^{(N+1)}(x)| \leq B$. Note that $T_N(x) \to f(x)$ if and only if $\lim_{N\to\infty} |R_N(x)| = 0$.

Basic Divergence Test: If $a_n \nrightarrow 0$, then $\sum a_n$ diverges. If $\sum a_n$ converges, then $a_n \to 0$.

Proposition: For any $m > 1$, $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=m}^{\infty} a_n$ both converge or both diverge.

Geometric Series: We have $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$. The series diverges if $|x| > 1$.

Integral Test: Let $a_n \geq 0$. for $n \geq 1$. Let $f(n) = a_n$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{\infty} f(x)\, dx$ converges.

P-Series: We have that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Error Estimates for Series with Nonnegative Terms: Let $\sum a_n$ be a convergent series and let $a_n \geq 0$ for $n \geq 1$ and $U_N = S_N + \int_{N+1}^{\infty} f(x)\, dx$. Then $0 < S - U_n < a_{N+1}$. In other words U_N is the area corrected approximation to S and $S - U_N$ is a good approximation to $|S - S_N|$ the error between the exact sum and the Nth partial sum.

Basic Comparison Test: Let $0 \leq a_n \leq b_n$ for all $n \geq 1$. Then

$$\sum b_n \text{ converges} \quad \Rightarrow \quad \sum a_n \text{ converges}$$
$$\sum a_n \text{ diverges} \quad \Rightarrow \quad \sum b_n \text{ diverges.}$$

Limit Comparison Test: If $a_n, b_n \geq 0$ for all large n and $\lim_{n\to\infty} \frac{a_n}{b_n}$ is positive (nonzero) and finite, then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Basic Conventions of Alternating Series: An alternating series can be written in the form

$$\sum_{n=1}^{\infty}(-1)^{n+1}b_n \quad \text{or} \quad \sum_{n=1}^{\infty}(-1)^{n}b_n$$

where $b_n \geq 0$. We will use this convention in the topics involving alternating sequences below.

Alternative Series Test: If the numerical size of the terms in an alternating series decrease and tend to zero (i.e. $b_n \to 0$), then the series converges.

Error Estimates for Alternating Series: If a series converges by the alternating series test, then $|S - S_N| < b_{N+1}$.

Absolute Convergence: A series $\sum a_n$ is said to be absolutely convergent if $\sum |a_n|$ is convergent. We have that if $\sum |a_n|$ converges, then $\sum a_n$ converges.

Root Test: Suppose that $|a_n|^{1/n} \to L$ (finite or infinite). Then $\sum a_n$ converges absolutely if $L < 1$, $\sum a_n$ diverges if $L > 1$, and the test fails (does not imply anything) if $L = 1$.

Ratio Test: Let $a_n \neq 0$ for all large n and $\left|\frac{a_{n+1}}{a_n}\right| \to L$ (finite or infinite). Then $\sum a_n$ converges absolutely if $L < 1$, $\sum a_n$ diverges if $L > 1$, and the test fails if $L = 1$.

Stirling's Formula: We have

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}(n/e)^n} = 1.$$

In other words, we may exchange $\sqrt{2\pi n}(n/e)^n$ for $n!$ in any limit as n goes to infinity.

Error Estimates Using the Ratio Test: Assume $\frac{|a_{n+1}|}{|a_n|} \leq r < 1$ for $n \geq M$. If $N \geq M$ then the partial sum S_N and sum S of the series $\sum_{n=0}^{\infty} a_n$ satisfy

$$|S - S_N| \leq \frac{|a_{N+1}|}{1-r}.$$

Error Estimates Using the Root Test: Assume $|a_n|^{1/n} \leq r < 1$ for $n \geq M$. If $N \geq M$ then the partial sum S_N and sum S of the series $\sum_{n=0}^{\infty} a_n$ satisfy

$$|S - S_N| \leq \frac{r^{N+1}}{1-r}.$$

Special Limits: As $n \to \infty$:

$$\begin{aligned} a^n &\to 0 \quad \text{if } |a| < 1 \\ a^{1/n} &\to 1 \quad \text{if } a > 0 \\ \frac{\ln(n)}{n^p} &\to 0 \quad \text{for } p > 0 \\ n^{1/n} &\to 1 \\ \left(1 + \frac{x}{n}\right)^n &\to e^x \quad \text{for all x} \\ \frac{x^n}{n!} &\to 0 \quad \text{for all x} \end{aligned}$$

Radius of Convergence: Let $\sum c_n(bx-a)^n$. Let $\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty}|c_n|^{1/n} = L$. Then the power series converges absolutely for

$$\left|x - \frac{a}{b}\right| < \frac{1}{b\sqrt[k]{L}}.$$

In other words the power series has center a/b and radius $\frac{1}{b\sqrt[k]{L}}$.

Magnitude of a Vector: Let $\mathbf{u} =< u_1, u_2, \ldots, u_n >$. Then the magnitude of $\mathbf{u}$ is $|\mathbf{u}| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$.

Dot Product: Let $\mathbf{u} =< u_1, u_2, \ldots, u_n >$ and $\mathbf{v} =< v_1, v_2, \ldots, v_n >$. Then $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$, where θ is the angle between the two vectors.

Perpendicular (orthogonal) Test: Two vectors are orthogonal if their dot product is zero.

Vector Projections: The vector projection of $\mathbf{b}$ along $\mathbf{a}$ is $\text{proj}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a}$.

Size of a Matrix: If a matrix, A, has n rows and m columns, we say that A is an $n \times m$ matrix.

Linear Equations: The set of linear equations

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

can be expressed in the form $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}.$$

We often solve the system by the augmented matrix $[A \mid \mathbf{b}]$ and using elementary row operations (or rref on our calculator) to reduce it to echelon form.

Determinate of a 2×2 Matrix: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $\det(A) = ad - bc$.

Inverse of a Matrix: For the inverse of a matrix to exist, it must be square and have nonzero determinate. To find the inverse row reduce the augmented matrix $[A \mid I]$, where I is the identity matrix of 1's down the diagonal and 0's everywhere else. The matrix B is the inverse of the matrix A if and only if $AB = BA = I$. We denote B by A^{-1}. Note that if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Span of a Set of Vectors: Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be vectors in $\mathbb{R}^n$. Then the span of this set of vectors is the subspace of $\mathbb{R}^n$ defined by $\{x \in \mathbb{R}^n : x = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots c_k\mathbf{v}_k$ for scalars $c_1, c_2, \ldots, c_k\}$.

Linear Dependence and Independence in $\mathbb{R}^n$: A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly independent (LI) if and only if the equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = 0$ is only true if $c_1 = c_2 = \cdots = c_k = 0$. If the above equation is satisfied with least one of the c_i's not equal to zero, then the set is linearly dependent (LD).

Test for Linear Independence: Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be vectors in $\mathbb{R}^n$ and $A = [\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_n]$. Then the vectors are linearly independent if $\det(A) \neq 0$, otherwise the vectors are linearly dependent.

More on Linear Dependence: Suppose that the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in $\mathbb{R}^n$ are linearly dependent. To find the $c_1, c_2, \ldots, c_n$ such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = 0$, solve the equation $A\mathbf{c} = 0$ by the method of row reductions where $A = [\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_n]$.

Matrices and Linear Transformations: See lesson 16.

Eigenvalues and Eigenvectors of a Matrix: We say that $\lambda \in \mathbb{R}$ is an eigenvalue of the matrix A if there exists a nonzero vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$. We also say that $\mathbf{v}$ is an eigenvector of A corresponding to the eigenvalue λ.

Determination of Eigenvectors and Eigenvalues: To find the eigenvalues of a matrix solve the equation $\det(A - \lambda I) = 0$. Since $\det(A - \lambda I)$ is a polynomial, this is logically equivalent to factoring a polynomial with variable λ. The roots of the polynomial are the eigenvalues of A. Now suppose that λ_1 is an eigenvalue of A. To find the eigenvectors of A corresponding to the eigenvalue λ_1 use row reduction techniques on the augmented matrix $[A - \lambda_1 I \mid \mathbf{0}]$.