Math 256 Differential Equations - Theorems and Definitions

Notation: First we describe the notation used below. Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{C}$ denote the set of complex numbers. Read $\in$ as "is an element of". For instance, the statement "2 is a real number" can be written $2 \in \mathbb{R}$. Every time an "i" appears it should be interpreted as $\sqrt{-1}$, the imaginary number. Let f_x denote the partial derivative $\partial f/\partial x$ and $y^{(n)}$ denote the n-th derivative of y. Also c_1, c_2 will always denote constants.

Differential Equation: A differential equation is an equation involving the derivative of one or more functions.

Linear Differential Equation: A linear differential equation is a differential equation of the form

$$a_0(t)\frac{d^n y}{dt^n} + a_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_n(t)y = g(t).$$

In other words, if you can't put the differential equation in this form, then it is a nonlinear differential equation.

Initial Value Problem (IVP): An initial value problem is a differential equation along with initial values of the unknown function.

Separation of Variables: Suppose that we are given a differential equation that we can write in the form $\frac{dy}{dx} = f(x)g(y)$. Then $\frac{dy}{g(y)} = f(x)\,dx$, so $\int \frac{dy}{g(y)} = \int f(x)\,dx$.

Integrating Factors: Suppose that $y' + p(t)y = g(t)$. Let

$$\mu(t) = e^{\int p(t)\,dt}.$$

Then

$$y(t) = \frac{1}{\mu(t)} \int \mu(t)g(t)\,dt.$$

Existence and Uniqueness Theorem for First Order Linear Differential Equations: If the functions p and g are continuous on an open interval $I = (\alpha, \beta)$ containing the point t_0, then there exists a unique function $y = \varphi(t)$ that satisfies the IVP $y' + p(t)y = g(t)$, $y(t_0) = y_0$.

Existence and Uniqueness Theorem for First Order Nonlinear Differential Equations: Suppose that the function $f(t, y)$ is continuous in some rectangle $R : \alpha < t < \beta$, $\gamma < y < \delta$ containing the point (t_0, y_0). Then, in some interval $I : t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there exists a solution $y = \varphi(t)$ of the IVP $y' = f(t, y)$, $y(t_0) = y_0$. If, in addition, $\partial f/\partial y$ is continuous on R, then there is a unique solution to the above IVP some interval on the interval I.

Exact Equations: Let M, N, M_y, and N_x be continuous in the rectangular region R: $\alpha < x < \beta$, $\gamma < y < \delta$. Then

$$M(x, y) + N(x, y)y' = 0$$

is an exact differential equation in R if and only if $M_y(x, y) = N_x(x, y)$. Then there exists a function $\Psi(x, y)$ such that

$$\Psi_x(x, y) = M(x, y), \qquad \Psi_y(x, y) = N(x, y)$$

and the solution to the differential equation is $\Psi(x, y) = c$. Then

$$\Psi(x, y) = \int M(x, y)\, dx + h(y),$$

where $h(y)$ is a function such that

$$h'(y) = N(x, y) - \frac{\partial}{\partial y}\left[\int M(x, y)\, dx\right].$$

Euler's Method: Euler's method can be used to approximate solutions to the IVP $\frac{dy}{dt} = f(t, y)$, $y(t_0) = y_0$. Then $y_{n+1} = y_n + f(t_n, y_n)(t_{n+1} - t_n)$ for $n = 0, 1, 2, \ldots$. In other words $y_n \approx y(t_n)$.

Picard Iteration: Suppose that $y' = f(t, y)$ and $y(0) = 0$. Let $\varphi_0(t) = 0$. Then Picard's Iteration is

$$\varphi_{n+1}(t) = \int_0^t f(s, \varphi_n(s))\, ds.$$

Under certain conditions on f, $\lim_{n\to\infty} \varphi_n(t) = y(t)$, where $y(t)$ is the solution to the given IVP.

Euler's Equation: We have $e^{\lambda + i\mu} = e^{\lambda}(\cos(\mu) + i\sin(\mu))$.

Wronskian: We denote the Wronskian of y_1 and y_2 by $W(y_1, y_2)$. Then

$$W(y_1, y_2)(t) = \det\left(\begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix}\right) = y_1(t)y_2'(t) - y_2(t)y_1'(t).$$

If $W(y_1, y_2)(t) \neq 0$ for some t, then y_1 and y_2 are linearly independent. Otherwise (Wronskian is always zero) they are linearly dependent. Also, if there exists a constant k such that $ky_1(t) = y_2(t)$ for all t, then y_1 and y_2 are linearly dependent.

General Solutions to Homogeneous Equations with Constant Coefficients: Solve $ay'' + by' + cy = 0$. Let $C(r) = ar^2 + br + c = a(r - r_1)(r - r_2)$. Then the general solution is

$$\begin{array}{ll} r_1, r_2 \in \mathbb{R} \text{ and } r_1 \neq r_2 & y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \\ r_1, r_2 \in \mathbb{R} \text{ and } r_1 = r_2 & y(t) = c_1 e^{r_1 t} + c_2 t e^{r_1 t} \\ r_1, r_2 \in \mathbb{C} \text{ and } r_1 = \lambda + i\mu, r_2 = \lambda - i\mu & y(t) = c_1 e^{\lambda t}\cos(\mu t) + c_2 e^{\lambda t}\sin(\mu t). \end{array}$$

Particular Solutions to Nonhomogeneous Equations; Method of Undetermined Coefficients(1): Let $L[y] = ay'' + by' + cy = q(x)e^{\gamma x}$ with $\deg(q) = n$ and $C(r) = ar^2 + br + c = a(r - r_1)(r - r_2)$. Then $p(x)e^{\gamma x}$ is a particular solution, where

$$\begin{array}{ll} p(x) = A_0 + A_1 x + \cdots + A_n x^n \text{ if} & C(\gamma) \neq 0 \\ p(x) = A_0 x + A_1 x^2 + \cdots + A_n x^{n+1} \text{ if} & C(\gamma) = 0, C'(\gamma) \neq 0 \\ p(x) = A_0 x^2 + A_1 x^3 + \cdots + A_n x^{n+2} \text{ if} & C(\gamma) = C'(\gamma) = 0. \end{array}$$

Then we have that $ap''(x) + C'(\gamma)p'(x) + C(\gamma)p(x) = q(x)$. We use the previous equation to solve for $A_0, A_1, \ldots, A_n$. Note that this works even when the roots of $C(r)$ are complex.

Particular Solutions to Nonhomogeneous Equations; Method of Undetermined Coefficients(2): Let $L[y] = ay'' + by' + cy = q(x)e^{\lambda x}T(x)$ where $T(x) = \cos(\mu x)$ or $\sin(\mu x)$ and $C(r) = ar^2 + br + c = a(r - r_1)(r - r_2)$. Let $\gamma = \lambda + \mu i$ and

$$\begin{array}{ll} p(x) = A_0 + A_1 x + \cdots + A_n x^n \text{ if} & C(\gamma) \neq 0 \\ p(x) = A_0 x + A_1 x^2 + \cdots + A_n x^{n+1} \text{ if} & C(\gamma) = 0, C'(\gamma) \neq 0 \\ p(x) = A_0 x^2 + A_1 x^3 + \cdots + A_n x^{n+2} \text{ if} & C(\gamma) = C'(\gamma) = 0, \end{array}$$

where the coefficients of p satisfy $ap''(x) + C'(\gamma)p'(x) + C(\gamma)p(x) = q(x)$. If $T(x) = \cos(\mu x)$, then a particular solution to $L[y] = q(x)e^{\lambda x}T(x)$ is $y_p(t) = Re[p(x)e^{\gamma x}]$. If $T(x) = \sin(\mu x)$, then a particular solution to $L[y] = q(x)e^{\lambda x}T(x)$ is $y_p(t) = Im[p(x)e^{\gamma x}]$.

Particular Solutions to Homogeneous Equations; Variation of Parameters: Let I be an interval such that $p(t)$, $q(t)$, and $g(t)$ are continuous. Let $L[y] = y'' + p(t)y' + q(t)y = g(t)$ and let $y_1(t)$ and $y_2(t)$ be linearly independent solutions to the homogeneous equation $L[y] = 0$. Then a particular solution to the nonhomogeneous equation $L[y] = g(t)$ is

$$Y(t) = -y_1(t)\int_{t_0}^{t} \frac{y_2(s)g(s)}{W(y_1, y_2)(s)}\,ds + y_2(t)\int_{t_0}^{t} \frac{y_1(s)g(s)}{W(y_1, y_2)(s)}\,ds,$$

where $t_0 \in I$.

General Solutions to Nonhomogeneous Equations: Let $L[y]$ be a linear second order differential equation. Suppose that y_1 and y_2 are linearly independent solutions to the homogeneous equation and y_p is a particular solution to the nonhomogeneous equation. Then $y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$ is the general solution to the nonhomogeneous equation.

Euler Equations: Let $L[y] = x^2y'' + \alpha xy' + \beta y = 0$ for $x > 0$ and $F(r) = r^2 + (\alpha - 1)r + \beta = (r - r_1)(r - r_2)$. Then the general solution is

$r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$	$y(t) = c_1x^{r_1} + c_2x^{r_2}$
$r_1, r_2 \in \mathbb{R}$ and $r_1 = r_2$	$y(t) = (c_1 + c_2 \ln x)x^{r_1}$
$r_1, r_2 \in \mathbb{C}$ and $r_1 = \lambda + i\mu, r_2 = \lambda - i\mu$	$y(t) = c_1x^\lambda \cos(\mu \ln x) + c_2x^\lambda \sin(\mu \ln x)$

for $x > 0$. To find particular solutions to nonhomogeneous Euler equations use the method of variation of parameters.

Mechanical Vibrations: Let m = mass, γ = damping coefficient, and k = spring constant.

Undamped Free Vibrations: Let $mu'' + ku = 0$. Then

$$u(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) = R\cos(\omega_0 t - \delta),$$

where $\omega_0^2 = k/m$, $R = \sqrt{A^2 + B^2}$, and $\delta = \arctan(B/A)$. Then ω_0 is the natural frequency, R is the amplitude, and δ is the phase.

Damped Free Vibrations: Let $mu'' + \gamma u' + ku = 0$. Let δ and R be as above and $\mu = \frac{\sqrt{4km-\gamma^2}}{2m}$. Then $C(r) = mr^2 + \gamma r + k = m(r - r_1)(r - r_2)$ and

$u(t) = Ae^{r_1t} + Be^{r_2t}$	if $\gamma^2 - 4km > 0$	(overdamped)
$u(t) = (A + Bt)e^{-\gamma t/2m}$	if $\gamma^2 - 4km = 0$	(critical damping)
$u(t) = e^{-\gamma t/2m}R\cos(\mu t - \delta)$	if $\gamma^2 - 4km < 0$	(underdamped).

We call μ the quasi frequency and $T = \frac{2\pi}{\mu}$ the quasi period.

Laplace Transform: The Laplace transform of a function f is given by

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st}f(t)\,dt.$$

Note that $\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$ i.e. $\mathcal{L}$ is linear.

Theorem 6.2.1: Suppose that f is continuous and f' is piecewise continuous on any interval $0 \leq t \leq A$. Suppose further that there exists constants K, a, M such that $|f(t)| \leq Ke^{at}$ for $t \geq M$. Then $\mathcal{L}\{f'(t)\}$ exists for $s > a$, and moreover

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0).$$

Theorem 6.2.2: Suppose that the functions $f, f', \ldots, f^{(n-1)}$ are continuous and that $f^{(n)}$ is piecewise continuous on any interval $0 \leq t \leq A$. Suppose further that there exists constants

K, a, M such that $|f(t)| \leq Ke^{at}, |f'(t)| \leq Ke^{at}, \ldots, |f^{(n-1)}(t)| \leq Ke^{at}$ for $t \geq M$. Then $\mathcal{L}\{f^{(n)}(t)\}$ exists for $s > a$ and is given by

$$\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

Solve Differential Equations with the Laplace Transform: Below is a partial list of Laplace and inverse Laplace transforms of some elementary functions. See the table on page 319 of your book for a more complete list.

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}, \quad s > 0$
e^{at}	$\frac{1}{s-a}, \quad s > a$
t^n	$\frac{n!}{s^{n+1}}, \quad s > 0$
$\sin(at)$	$\frac{a}{s^2+a^2}, \quad s > 0$
$\cos(at)$	$\frac{s}{s^2+a^2}, \quad s > 0$
$u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$

Heaviside Function: The Heaviside function is given by

$$u_c(t) = \begin{cases} 0, & t < c, \\ 1, & t \geq c, \end{cases} \qquad c \geq 0.$$

Theorem 6.3.1: If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$ and if $c \geq 0$, then

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s)$$

for $s > a$. Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$u_c(t)f(t-c) = \mathcal{L}^{-1}\{e^{-cs}F(s)\}.$$

Theorem 6.3.2: If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$, and if $c \in \mathbb{R}$, then

$$\mathcal{L}\{e^{ct}f(t)\} = F(s-c),$$

for $s > a + c$. Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$e^{ct}f(t) = \mathcal{L}^{-1}\{F(s-c)\}.$$

Convolutions: The convolution of f and g is given by

$$f * g(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau.$$

Note that $f*g = g*f$, $f*(g_1+g_2) = f*g_1 + f*g_2$, $(f*g)*h = f*(g*h)$, and $f*0 = 0*f = 0$.

Theorem: If $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$ both exist for $s > a \geq 0$, then

$$H(s) = F(s)G(s) = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = \mathcal{L}\{f * g(t)\} \quad s > a.$$

The above contains notes on about everything covered this term except for direction fields (section 1.1) and modeling (section 2.3).